Mathematical Analysis - List 7

1. Find the vertical and oblique asymptotes of each function:

a) $f(x) = \dfrac{x^3 + x^2}{x^2 - 4}$; b) $f(x) = \dfrac{x - 3}{\sqrt{x^2 - 9}}$; c) $f(x) = \dfrac{\sin x}{x - \pi}$;

d) $f(x) = \dfrac{\sqrt{1 + x^2}}{x}$; e) $f(x) = \dfrac{x^3}{(x + 1)^2}$; f) $f(x) = \dfrac{1 - x^2}{x + 1}$.

2. Find numbers $a, b \in \mathbb{R}$ such that the function $f(x)$ is continuous at the given points.

a) $f(x) = \begin{cases} \sin x & \text{for } |x| \geqslant \dfrac{\pi}{2}, \quad x_1 = -\dfrac{\pi}{2}, \\ ax + b & \text{for } |x| < \dfrac{\pi}{2}, \quad x_2 = \dfrac{\pi}{2}; \end{cases}$

b) $f(x) = \begin{cases} x^2 + ax + b & \text{for } |x| < 2, \quad x_1 = -2, \\ x\sqrt{x^2 - 4} & \text{for } |x| \geqslant 2, \quad x_2 = 2; \end{cases}$

c) $f(x) = \begin{cases} bx^2 + a & \text{for } x \leqslant 0, \\ \dfrac{5^x - 3^x}{ax} & \text{for } x > 0, \quad x_0 = 0. \end{cases}$

3. Find the points at which the function is discontinuous.

a) $f(x) = \begin{cases} \dfrac{x^2 - 1}{\sqrt{x} - 1} & \text{for } x \in [0, 1) \cup (1, \infty), \\ 3 & \text{for } x = 1; \end{cases}$

b) $f(x) = \begin{cases} \dfrac{|x| + x}{x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0; \end{cases}$

c) $f(x) = \operatorname{sign}\left[x(x - 1)\right]$;

d) $f(x) = \begin{cases} 1 - \cos\dfrac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$

4. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

a) $x^3 + 6x - 2 = 0$, $(0, 1)$; b) $x \sin x = 7$, $\left(2\pi, \dfrac{5\pi}{2}\right)$;

c) $1 = \dfrac{\sin x}{2} + x$, $\left(0, \dfrac{\pi}{2}\right)$; d) $x^{100} + x - 1 = 0$, $\left(\dfrac{1}{2}, 1\right)$.

Find the root in a) correct to two decimal places.